# A GENERALIZATION OF A THEOREM OF BOHR AND NEUGEBAUER

(OB ODNOM OBOBSHCHENII TEOREMY BORA I NEIGEBAUERA)

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Let the differential equation

$$z^{(n)} + a_1 z^{(n-1)} + \ldots + a_{n-1}z' + a_n z = f(t), \qquad z^{(k)} = \frac{d^k z}{dt^k} \tag{1}$$

be given, where $a_1, \ldots, a_n$ are constants and $f(t)$ is an almost periodic function. Bohr and Neugebauer [1] proved that any bounded solution of equation (1) is an almost periodic function. Consider the system

$$\frac{dx_s}{dt} = P_{s1}x_1 + \ldots + P_{sn}x_n + f_s(t) \qquad (s = 1, \ldots, n) \tag{2}$$

where $P_{sk}$ are periodic functions of the independent variable $t$ with a common real period and $f_s(t)$ is an almost periodic function in the sense of Bohr.

*Theorem.* Any bounded solution $x_1(t), \ldots, x_n(t)$ of the system (2) is a set of almost periodic functions.

In fact, since the homogeneous system corresponding to system (2) is reducible, then by means of two successive nonsingular transformations with periodic and constant coefficients, respectively, the system (2) is transformed into a system of equations which consists of several independently integrable groups of the form

$$\frac{dz_1}{dt} = \lambda_1 z_1 + \Psi_1, \qquad \frac{dz_2}{dt} = z_1 + \lambda_1 z_2 + \Psi_2, \ldots, \qquad \frac{dz_{P_1}}{dt} = z_{P_1-1} + \lambda z_{P_1} + \Psi_{P_1} \tag{3}$$

The functions $\Psi_s(t)$ are almost periodic. Let $x_1(t), \ldots, x_n(t)$ be a certain bounded solution of system (2). Submitting this solution, successively, to the above mentioned transformations, we obtain a bounded solution of a system, consisting of groups of the form (3). By the same token, on the basis of the results obtained by Bohr and Neugebauer, the theorem is proved.

*Remark 1.* On the basis of the theorem just proved the theorem of Bohr and Neugebauer can be expressed in a more general form as follows: any bounded solution of equation (1), where $a_1, \ldots, a_n$ are either constants or periodic functions of the independent variable $t$ with a common real period, is an almost periodic function.

*Remark 2.* Assume that the homogeneous system corresponding to system (2) has no zero characteristic numbers. Then Re $(\lambda_j) \neq 0$ and, as seen from (3), the system (2) assumes but one unique almost periodic solution. This solution satisfies the inequality

$$|x_s(t)| \leqslant \frac{P\Gamma}{\gamma^n} \qquad (\Gamma = M_1 + \ldots + M_n, \quad M_s \geqslant |f_s(t)|) \tag{4}$$

where $P$ is a certain constant, depending only on the matrix $\| P_{sk}(t) \|$, and $\gamma$ is a positive constant satisfying the condition $|\mathrm{Re}(\lambda_j)| > \gamma$. In fact, we have

$$|\Psi_s(t)| \leqslant A(M_1 + \ldots + M_n) = R$$

where $A$ is a quantity which depends on the coefficients of the two successive nonsingular transformations. Considering the group of equations (3), we have the estimations

$$|z_1| \leqslant \frac{R}{|\lambda_1|}, \ldots, \qquad |z_{P_1}| \leqslant \frac{R(1 + |\lambda_1| + \ldots + |\lambda_1|^{P_1 - 1})}{|\lambda_1|^{P_1}}$$

Analogous estimates can be obtained for any other group of equations. Putting $z_s$ successively (in the reversed order) to the above transformations we obtain inequality (4).

BIBLIOGRAPHY

1. Bohr, H. and Neugebauer, O., Uber lineare Differentialgleichungen mit konstanten Koeffizienten und fast periodischer rechter. *Seite. Gott. Nachr.*, 1926.

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